Supporting Teacher Proving Practices with Three Phases of Proof

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Muhammed Fatih Dogan
Adiyaman University
and
Caro Williams-Pierce
University at Albany, SUNY (now at the University of Maryland, College Park)

Abstract
Although reasoning and proof in learning and teaching mathematics is crucial and have gained more presence in school mathematics, both students and their teachers face great difficulties when engaging in proving activities. One potential cause for such difficulties might be due to teachers' conception of proof. However, to date, there are few, if any, studies that have examined how secondary school in-service mathematics teachers learn justification and proof. This study focuses on secondary school in-service teachers' engagement in proving activities by providing observational data from a master's level professional development course that focuses on teaching reasoning and proof. The findings from this work highlight the usefulness of framing proving activities as consisting of three phases: exploration; justification; and evaluation. In addition, we discuss the useful role that generic example-based proofs can play when teachers are proving. We illustrate through a specific vignette of teacher proving activities, and discuss the results in the light of using proof-related tasks to engage learning during proving activities.

Keywords mathematics teacher education, professional development, reasoning, justification, exploration, evaluation, generic proof.

Introduction
The importance of proof in teaching and learning mathematics is unquestionable with research on reasoning, justification, and proof is growing rapidly across the globe (Ball et al., 2002; Hanna, 2018; Healy & Hoyles, 2000; Reid & Knipping, 2010). Reform documents and mathematics educators in various countries suggest that proof is a crucial activity that mathematics students of all ages should participate in on a regular and consistent basis (e.g. the *Principles and Standards for School Mathematics* by the National Council of Teachers of Mathematics [NCTM], 2000; the Common Core State Standards for Mathematics by CCSSO, 2010; Department for Education, 2013; Conner, 2013; Healy & Hoyles, 2000; Reid & Knipping, 2010). As Ellis (2011:526) noted, 'Providing regular opportunities for students to prove and explicitly supporting students' emerging proof abilities will help middle school students not only become more adept at proving but also develop a deeper understanding of the mathematics they investigate'. Yet, research indicates that students at all levels struggle to understand and construct proofs (e.g. Ellis et al., 2017; Ozgur et al., 2019; Chazan, 1993; Harel & Sowder, 1998; Weber, 2005), and that teachers often have difficulty fostering students' ability to justify and prove (Knuth, 2002; Bieda, 2010). Indeed, it is not a secret that the concept of proof is found to be problematic from both teaching and learning points of view (Stylianides, Stylianides, & Weber, 2017) as it is difficult for students to grasp and hard for teachers to teach. However, there is limited research on the role of proof in ordinary mathematics classrooms at K-12 levels and the place of proof in these classroom practices. Therefore, this paper aims to discuss how to support teachers to teach proof by illustrating an example of teachers' own engagement on proving activities.

The context for this study is a professional development course about justification and proof for in-

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service secondary teachers in the United States of America. These teachers primarily teach students ranging from 11 to 15 years of age. Proof in the United States has historically been restricted to high school and undergraduate geometry courses, although there is a growing movement to infuse justification and proof throughout all mathematics courses starting in the first grade (NCTM, 2000; CCSSO, 2010).

Background and The Three Phases of Proof

Our definition of proof is one that may be unfamiliar to some, as we follow Stylianides' (2007) definition, which we define in more detail below. Our approach includes reasoning and justification as crucial components of the proving process - that is, our view of proof is not limited to deductive proving. In particular, one key aspect is that a learner engaging in proof is engaging in convincing oneself and convincing others (e.g., Harel & Sowder, 1998) - consequently, a learner merely duplicating the instructor's deductive written work is not engaging in proof as we define it. The important aspect of teaching proof, in our view, is the focus on teaching reasoning and justification, as considering deductive proof as the primary goal of teaching proving can result in negative effects for learning and reasoning (Duval, 1998, 2007; Kuchemann, 1981).

To that end, we frame proof as a mathematical activity with a social character 'as a means of convincing oneself whilst trying to convince others' (Alibert & Thomas, 1991:215), which allows learners to engage in different phases of proof production, such as task exploration, producing justification and proof, and evaluating arguments as mathematics classroom communities. Thus, mathematical proof is seen as an argument that consists of a connected sequence of assertions about a claim *and* a norm for communicating argument, in which what counts as acceptable proof is determined by the classroom community. Stylianides (2007:291) provides a definition of proof that aligns with mathematicians' conception of proof and gives emphasis to the social aspects of students' proving activities in school mathematics.

> *Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:*
> (1) *It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification;*
> (2) *It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and*
> (3) *It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community.*

Seen in this way, proof is a tool for teaching mathematics that engages students in meaningful and rich mathematical activities (Bieda, 2010; Stylianides, 2007), and consequently plays an important role in promoting deep learning of mathematics (Ellis, 2011; Hanna, 2000). In order to explore the complexities of classroom practices related to proof, we discuss three phases of proving related activities: exploration and conjecturing, justification and proof, and evaluation phases (Dogan, 2015, 2019).

Exploration

A proving activity often begins with students developing conjectures by making observations about mathematical problems (Dogan, 2015, 2019). When developing conjectures about mathematical problems, students work to remove or create doubts about the conjecture in order to convince themselves and others in the classroom community about the truth of the conjecture. This process is called the *exploration phase* of the proving activity. At this point students may or may not produce an argument that can be considered a proof, but the process of exploring allows them to gain insights about the problem and uncover important features of the mathematical conjecture. However, the next step – moving from exploration to justification – may need to be prompted. Research has shown that students and teachers often do not produce a justifications and/or proofs without being prompted to do so (e.g. Bieda, 2010; Dogan, 2019). Adding the prompts such as *'Why?'* or '*How do you know?'* often can help students to consider producing a justification or proof, and moving to the second phase in the proving process.

Justification and Proof

In this phase, students justify their arguments and explain their reasoning (Dogan, 2015, 2019). This process does not necessarily lead students to develop a formal mathematical proof, but it allows them to explicate mathematical ideas and to communicate with others in a mathematical way by developing an argument, requiring their active participation in the proving activities. This process is called the *justification phase* of proving. Students may present final or semifinal proofs during this justification phase.

Evaluation

Proving activities do not end with the justification phase. Rather, the community must then decide if a justification counts as a valid mathematical argument or not. This phase is called the *evaluation phase* and is a crucial part of proving activity because it is when the validity of the argument is determined, and also provides an opportunity to clarify important mathematical principles (Lannin, Ellis, & Elliott, 2013). The evaluation phase is a time for the prover and their community to reflect on the whole process of proving activity by determining whether a justification includes 'correct or mistaken assumptions, valid conclusions with erroneous logic, or valid arguments that nevertheless explain only portion of the statement' (Lannin et al., 2013:45).

We believe that having opportunities for learners of mathematics to engage in all three phases of proving activity is crucial not only for their learning of mathematics but for teaching mathematics in a meaningful way. Thus, framing mathematical practices and proving activities around these three phases might contribute to overcome some of the challenges in learning and teaching proof, and provide rich learning opportunities with both the mathematical content and the proof activities for students. Even though we present these three phases as separate parts of a proving progression, they are deeply interconnected, and provers may, for example, return briefly to the justification or exploration phase while being evaluated by their peers.

The Context and Math Problem

This article focuses on an analysis of an episode from a professional development (PD) class at a large midwestern university in the United States of America that focused on justification and proof at the middle school level. Even though the excerpt discussed here is from a PD course, the nature of the activities and the task itself is reflective of middle school mathematics and students' learning. All data was collected ethically with prior approval from the university, and written consent by the instructor and the participants. It is important to note that we did not explicitly introduce the three phases of proving outlined above, as these phases are results that emerged directly from this data.

The way that tasks are presented can encourage students to produce justification and proof in a meaningful way, as 'students will be better poised to develop proofs if they encounter tasks that create the need for substantial mathematical reasoning' (Ellis, 2011:524). To that end the following task was presented in the PD class and teachers were asked to produce justifications for their arguments:

> Each element in the Fibonacci sequence is constructed by adding together the two previous numbers in the sequence.

1, 1, 2, 3, 5, 8 are the first 6 terms. Is the 20th term odd or even? Is the 500th term odd or even?

The instructor selected this task in order to support teachers engaging in mathematical reasoning and proof in meaningful way, and the results bear her selection out. We share a vignette below with this math problem being used to support justification and proof activities. In that vignette, Anne's group develops a particular representation of the above conjecture – in order to both illustrate the richness of this problem, and prepare the reader for Anne's description of their proof practices, we explain their approach to this problem using a formalised version of her response.

The marble machine (shown in Figure 1) allows us to see a definition of even and odd. An even number of marbles fills full rows of the 'machine,' because each row represents '2.' Alternatively, the marble machine shows that in an even number of marbles, each marble is matched with another and the columns are filled equally. Likewise, an odd number when put into the marble machine will not fill rows evenly. The top row will have one missing. Alternatively, we can think of an odd number as not having the same number of marbles in both columns.

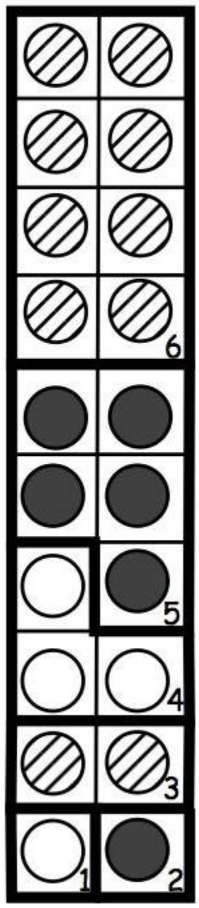

Figure 1. A formalised version of the Marble Machine.

So if we put in numbers from Fibonacci's sequence we see that:

(1) The first two terms are odd. Adding those two odd numbers together in the marble machine shows us that an odd plus an odd is even because the two odd numbers together complete a row. The third term will thus be even so we start with O, O, E
(2) The fourth term is generated from the two previous terms. So we are adding together the even number we just generated (2) to the previous term which was odd (1). We can see that

this fourth term will be odd because it does not fill rows completely because the even fills rows completely but the added odd number will not. Now we have O, O, E, O

(3) The fifth term is generated from the two previous terms. We have an odd (3) plus an even (2) which again yields an odd (5) by the same logic as in step 2. Now we have O, O, E, O, O.

(4) We know from step 1 that two odd numbers when added yield an even so now we are back to the beginning of the cycle, and we can see how it will repeat: O, O, E, O, O, E, and so on every three numbers. Based on this we can tell whether any term in the sequence will be even or odd by just dividing by 3. If it is a multiple of 3, the term will be even. If not, the term will be odd.

In the following section, we describe how Jane established the community norms for the proof activity, how Anne explored and justified her group's proof, and how both Anne and her classmates evaluated the proof. Finally, we conclude by describing how Anne's proof activities resulted in her development of a generic example.

Classroom Vignette

The instructor of the class, Jane (all names are pseudonyms), introduced the Fibonacci task and asked teachers to work on the task in groups. Before engaging in the task, Jane asked the teachers to define odd and even numbers, and she put all of the definitions shared on the board. By doing this, she established the set of accepted statements that are true and available without further justification (Stylianides, 2007) for the classroom community. Another characteristic of the proof community that Jane had established since the beginning of the class was that visual forms of proofs are valid mode of argumentation, following Hanna's (2000) conceptualisation of explanatory proofs. Specifically, visual representations can convey *why* a proof might be true, and 'proof can make its greatest contribution in the classroom only when the teacher is able to use proofs that convey understanding' (Hanna, 2000:7). We will discuss this further in a later section.

The teachers provided different kinds of definitions, including; 'an even number is a number that has at least two factors, one always being 2'; 'Even is when you can divide the group into couples, you know it's always into couples. And then the odd would be the opposite, it was always the third wheel'; 'it divides evenly by 2 (resulting in a whole number)…without a remainder'; and 'every other number... Ends in zero, two, four, six, eight… it's count by twos.' After these definitions, another teacher provided an algebraic definition (i.e., if *n* is an integer, then an even number is defined as 2*n*). Thus, the set of accepted statements that were discussed during the launch of the task provided the teachers with some of the tools they might need to construct a justification and proof.

After exploring the task in groups, Jane asks Ann to share her group's justification. Anne began by describing her exploration phase, interwoven with her justification phase since she is speaking to the whole class, and introduced a method of visualising the Fibonacci sequence which specifically attended to the definitions of even and odd numbers (Figure 2):

Anne: 'So I—I had this imagination like a marble tower where you put—you dump marbles in and it funnels them into—and there's, like, side-by-side. So, two side-by-side channels and you put the marbles in and then they'll—they'll stack up…Like a whole… so I imagined, like, okay, so I dump 7 marbles into my machine [referring to the left-hand of the Figure 2]. Then we get like 2, 2, 2, so three groups of 2—we've been talking a lot about how that works—and then one—it—the shell isn't filled there. Like, or, like, this part isn't filled. But then if I put in 5 green ones, then it'll—the one extra one will fill it in, and then—it dumps up. But if I put these purple ones in—if it—now I've got an even number here. Right? Because I added those. But then if I put in even ones it just fills it even—up even more, if that makes sense.'

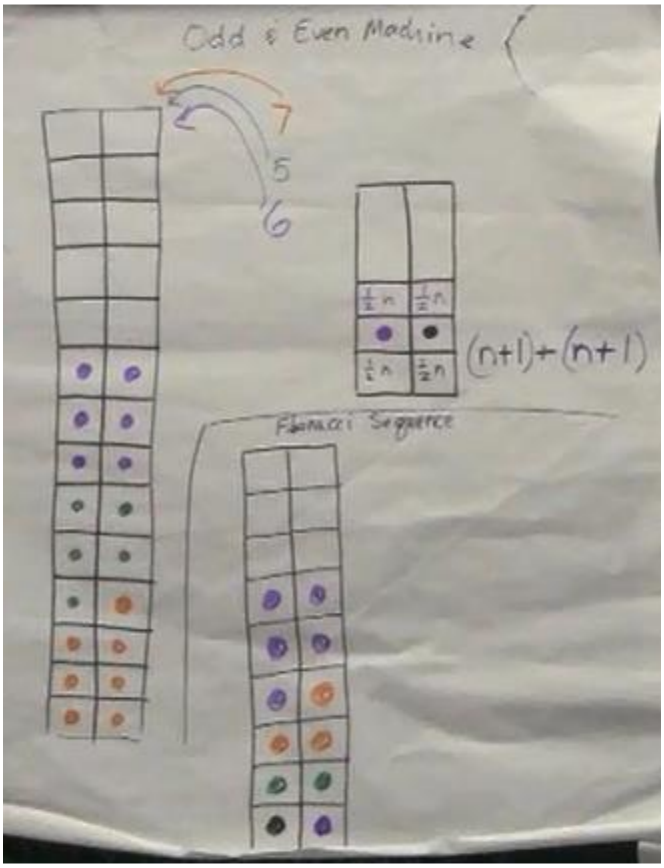

Figure 2. Anne's drawing of the Marble Machine.

The other teachers said her description of her exploration phase during her justification made sense to them and was easy to understand. For example, Carol said:

> 'I think it's—for me, like, it's the kind of—it's the most understandable instead of just a statement of saying that this is the pattern, you're actually—you can visualise the pattern and it, it reinforces and it also, like you said, it, it, it's that any number.'

Both teachers and the instructor accepted Anne's argument as a valid justification for adding even and odd numbers. This argument is communicated with an appropriate mode of representation (visuals and algebraic-symbolic notation). However, it is important to note that, for Anne, the visual did not seem like an entirely appropriate mode of representation, and consequently she continued to describe her struggle to generalise and justify her proof:

> Anne: 'And then I tried to think about how I could describe that with any number and…I thought, so, n plus 1 is our, uh, our stock term for what an odd number is. I thought, well, half of n would go here and half of n would go here, which I guess, kind of works.'
> Jane: '2n and 2n +1.'
> Anne: 'Yeah. I could think—I guess I could think of it as 2n plus 1. That might be better. So then it'd be n, n, and then plus one more dot. And then I thought—so then here's the extra dot from the other—from the other odd piece, if that makes sense.'
> Carol: 'I think the kids would actually—I think the kids would actually really get that—how the odd plus odd would stack up. And then meeting—and then you're getting an even. And if you add an even, it's stays even but if you add the odd, I mean, that becomes the Fibonacci sequence…As long as you continue to do it with—'

Anne: 'Yeah. Yeah. I was kind of looking there at doing it that way but I stopped there.'

After Anne presented her exploration phase, justification, and attempt to further generalise, Jane asked the class explicitly to evaluate Anne's argument. Some of the teachers provided the following responses.

> Jane: 'And this idea—how I take this and then say, like, for all the numbers. But if I have a larger odd number, what's it going to look like? Just looking at the orange.'
> Clara: 'It's going to have an extra one.'
> Anne: 'It's going to be filled up underneath however far it would be filled up and then there would be one that would be half empty.'
> Lora: 'So when you wrap--there's two in a container, right? So once you wrap them, they go away, right? And all you're left with is the odd one. So all of the even ones that match up—Those are taken away, right? …Well they're wrapped up and gone. And what's left is the one. Always. No matter—'
> Becca: 'It's always going to be 2n plus 1 though.'
> Jane:'—it's always going to be 2n—'
> Becca: 'Regardless.'
> Jane: '—plus one. So that kind of gets at the idea. You have specific numbers in terms of the—number of dots you have. … If the orange was just representing any odd number, we can just think of, well there's all that—like you guys said—there's all that stuff underneath that we can just kind of forget about—all of the—those pairs—that just keep going…We're getting into that any number even though we're looking at sort of a visual of that.'

At this point, teachers evaluated the argument based on if it made sense and tried to explain the structure behind Anne's argument. The teachers used general language that extends beyond a few examples. Another teacher, Becca, built on this discussion by using algebraic-symbolic representation and then Jane wrapped up the whole classroom discussion by summarising Anne's overall proof.

From the excerpts above, it is clear that teachers were meaningfully engaged in proving activities by exploring the task, producing justification and proof, and evaluating the arguments.

Generic Examples

Although her explanation above is actually more about experimenting with the 5 and 7 in the left-hand figure, Anne provided an argument called a generic argument (Balacheff, 1988). A generic example is a particular example that reveals the general structure of reasoning without relying on the properties of the particular example and reduces the level of abstraction to make the argument more accessible to learners. In other words, Anne began her justification with a few examples that she used to show the structure of adding even and odd numbers. Thus her justification does not rely on the particular example she describes but on a general structure of the task, which in this case was seeing the structure of odds being represented as a sum of repeated 2s, and then an additional 1. Furthermore, by providing a visual model, Anne's justification reduced the level of abstraction in a way that helps teachers see the structure of adding even and odd numbers.

It is interesting that Anne did desire to generalise to all numbers by representing with algebraic-symbolic representations even though she had a generic example argument and a visual model that could be used for any numbers (top right corner of Figure 2). This might be because she believes that proof always needs to have algebraic-symbolic notation, and therefore, she's not able to see her statement as generic. Another important point here is that she chose to represent odd numbers as n+1 and each whole (any number), as $\frac{1}{2}n$. This is mathematically correct but makes it harder for others to understand because $\frac{1}{2}n$ may not always be a whole number. (The way she defines odd numbers, though, ensures that $\frac{1}{2}n$ is always a whole number because n is an even number). Because of that confusion, another teacher suggested using 2n and 2n+1 instead of n+1, which were accepted as general representations of even and odd numbers by Anne. Furthermore, the task was about developing an argument for whether the 20^{th}, 500^{th} and n^{th} terms of the Fibonacci sequence were odd or even, but Anne first provided a valid mathematical argument about adding even and odd numbers that would help her to create a similar argument for the original task.

Discussion

In this section, we focus specifically on the establishment of community norms, and how that supported productive proof activities. Then we discuss generic proofs and their potentially powerful role in the proving process, and conclude by discussing how the three stages of proof can provide a useful way to frame classroom proving activity.

Establishing Community Norms and Prior Knowledge

The instructor's first step of establishing a set of accepted statements from which proof activities could emerge provided rich fodder for proving without needing to prove smaller components first. For example, Anne was able to develop a visual proof that relied upon the characteristics of even and odd numbers without worrying about using characteristics or representations that would not be accepted by the class as legitimate. We can see the results of Jane's explication of *what is known to be true* by the smoothness of Anne's justification – in other words, her classmates do not challenge her on the way that she represents and discusses odd and even numbers.

In addition, Anne used modes of argumentation and forms of expression that were established as legitimate in previous classes. This does not mean that the class – and indeed, Anne herself – consider her proof fully legitimate. Rather, despite the established mode of argumentation of visual proofs, and their preference for explanatory proofs (Hanna, 2000), they struggle with the idea of a visual proof being a complete proof, and are unsure whether the generic example is sufficiently general.

Generic Examples

Anne's use of generic examples parallels our findings with teachers using and evaluating generic examples in previous research (Dogan & Williams-Pierce, in submission). In particular, we found that teachers use three different types of generic examples: empirical arguments enriched by the use of generic examples; incomplete deductive arguments; and complete deductive arguments. Anne used an empirical argument infused with generic examples, which is a richer and more generalisable form than a purely empirical argument. We found that when teachers use generic proofs, they often conflate them with visual representations, and within that conflation, consider generic proofs (visual representations) best for teaching. However, they consider generic examples to be more convincing – and more complete of a proof – with symbolic notation alongside the visual form, just as Anne felt moved to include.

Framing Classroom Activity Productively

When traveling through the three phases of proof (exploring, proving and justifying, and evaluating), students and teachers are likely to test the next few examples in the sequence, note the even/odd pattern from the set of examples, and then make a generalisation about which terms of the sequence are odd and which are even (e.g. Dogan, 2015, 2019). Then, while presenting about the first two stages, the prover engages in the public process of having their proof evaluated, which can often lead the prover in revising their initial proof. As mentioned previously, these three phases of proof are complexly interwoven - however, we treat them separately here for simplicity's sake.

Exploring

Students can learn a great deal about justification and proof by just engaging in a task that requires them to produce a justification (Ball et al., 2002; Knuth, 2002; Stylianides, Stylianides, & Weber, 2017). But, it is not just matter of giving learners the tasks. Students must be allowed to devote a substantial amount of time and energy to the exploration of the task in order to develop a meaningful understanding and to produce a valid mathematical justification and proof (Bieda, 2010; Dogan, 2019). This not only would hold learners accountable to provide a general argument but would make proof a meaningful everyday practice in the learning of mathematics. For example, Anna used the whole class' definition of odd and even numbers to make sense of the conjecture, by using 7 marbles to show an odd number does not make group of 2, then adding 5 marbles to show adding to odd numbers makes even number-groups of 2, and finally adding 6 marbles to show adding even numbers keeps the groups 2s. In addition, during the exploration phase of proving activity, it is not uncommon for students to get stuck with the task and thus not be able to produce a justification or proof. Here, Anna had a similar experience while she tried to use the same strategy to generalise her idea to all numbers, and required getting help from the whole class in order to produce a justification for her claims. Thus, it is also important to push students to use and explain different representations, such as pictures, while engaging in the task (Stylianides, 2007). This helps students construct representations of the tasks that shows meaningful structures or patterns that can help them develop a valid justification and proof.

Justifying and Proving

After exploring the task and developing justifications and proofs, students need to have opportunities to present their arguments to the whole class (Conner, 2013). As seen from the episode above with Anna and feedback from the whole class, students often need to present their arguments in order to be provided opportunities to get feedback about their justification and proofs so that they can discuss what constitutes a mathematically appropriate justification and why particular justifications, such as example-based justifications, may not be sufficient for proof. Students also need to discuss why their arguments make sense and whether the arguments are convincing to themselves, their teachers, and most importantly their peers (Lannin, Ellis, & Elliott, 2013). Just presenting an argument may not guarantee that students understand what is happening and why that particular argument counts as proof. Thus, students need to be asked to present their argument and explain why the argument makes sense. This public proving phase leads to other members of the community in engaging with the evaluating phase.

Evaluating

Another critical classroom practice related to justification and proof is evaluating arguments. As seen from the episode above, Jane encourages teachers to evaluate the argument presented by Anne. Teachers explained how Anne's justification made sense to them and why it is a valid argument for adding even and odd numbers. In addition to encouraging students to share and evaluate the existing argument in the classroom, the students may be provided examples of hypothetical students' justifications that consist of both correct and incorrect justifications and be asked to evaluate those arguments (Lannin, Ellis, & Elliott, 2013). This might help students learn both what does and does not count as a mathematically valid argument, and thus contribute to their development of a deeper understanding of proof. Therefore, middle school teachers should emphasise what makes a particular justification valid and appropriate after letting their students discuss it.

Explicitly Introducing the Three Phases to Learners

As the three phases of proof are a finding that emerged directly from this research project, we did not explicitly introduce these phases as a tool to the participants. However, we believe that formally introducing these three phases to teachers in professional development, and supporting them in likewise introducing these three phases to their own students, would be beneficial. The three phases of proof may specifically help learners regulate their own proving and learning processes, as each particular phase of proving involves different but equally important goals (Dogan, 2019). We found that while our teachers explored conjectures and justified their proofs by convincing themselves and others, the third phase of evaluation was often minimised or quickly passed over. Formalising this phase as a key part of proof and justification may improve learning experiences and the quality of proofs.

Conclusion

While additional research is needed to examine how framing proof activities with the three phases for teacher professional development supports both teacher learning and the learning of their students, it is clear that the three phases are a useful tool for organising the otherwise deeply complex nature of proving. In particular, the three phases of proof activity – exploration, justification, and evaluation – are a useful and productive way to support mathematics teachers in engaging in authentic proof activities (Dogan, 2019). However, it is crucial that the professional development facilitator establish community proving norms prior to the proof activity, including both mathematical definitions and viable modes of argumentation. Generic proofs may be a particularly useful form of proof to support, as they can satisfy the teachers' desires to have both general arguments and explanatory proofs for their students (Dogan & Williams-Pierce, in submission).

Our future research goals are to examine any benefits that students and teachers may gain when using the three phases of proof to organise their engaging in and teaching of proof. In particular, we are curious as to how using these three phases of proof in teacher professional development may decrease the emphasis these teachers have inherited from their own educational experiences on deductive proof, and may increase the importance of both exploring and evaluating within the proving process.

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