Production of a hands-on learning resource to teach the topic of loci at GCSE

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Abstract

The origin of this study is rooted in an interest to design an innovative resource to teach the GCSE Mathematics topic of Loci in a more engaging and contextualised way. The aim of the resource is to provide pupils, of differing learning styles, with the opportunity to learn Mathematics through a range of mediums, particularly through hands-on and contextualised learning. The effectiveness of producing a hands-on learning resource to teach the GCSE topic of Loci is the main research question considered in this study. The planning, design and creation of the resource consisted of two key elements; a set of five hands-on learning boards, each relating to the five specific concepts of Loci, and a Prezi Presentation, *'5 Loci in Locustown'* which will guide both pupils and teachers around the tasks. This resource is also accompanied by a teacher-guide, acetate overlays and counters.

The resource was trialled with two experienced GCSE Mathematics teachers, four undergraduate Mathematics teachers and two GCSE Mathematics pupils. Qualitative and quantitative data, was collected for research purposes through the conduction of semi-structured interviews and completion of web-based questionnaires.

The results in the interviews and questionnaires highlighted that the resource was very successful in engaging students in the topic of Loci through the implementation of hands-on learning and ICT. However, some respondents commented that perhaps more time could have been spent trialling the product and developing it further to ensure a deeper understanding of the hidden Mathematics.

The research project provided the opportunity to be creative in designing an innovative resource to teach the GCSE topic, *Constructing Simple Loci*. Like many other Mathematical topics*,* Loci can be exciting when incorporating hands-on learning and ICT and made relevant to pupils when problems are contextualised. This way of teaching avoids isolation within learning and ensures that a pupil's learning in Mathematics is linked to their real-life experiences.

Based on these key findings and taking account of the strengths and limitations of this research, the project could be further developed to ensure there is full understanding of the underlying Mathematics. More research needs to be invested into how pupils of all abilities and learning styles can benefit from using manipulatives in the classroom and relating Mathematics to real life situations.

Introduction

> Mathematics is fundamental to life in the sense that its unique language and forms of notation help us to calculate, estimate and problem-solve. It also informs many of the choices and decisions we make about real-life issues and challenges and the actions that we subsequently take
>
> (CCEA, 2016).

Citation

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Despite most of the world's practical Mathematics aiming to solve problems in finance, science or design, many pupils believe that the 'word problems' in textbooks and standardised tests are very disconnected from their lives. Therefore, mathematics should be presented using real life contexts. McREL (2012, p.56) believes that if mathematics is taught in 'rich and realistic contexts, rather than on a purely abstract basis,' more students are able to build a deeper understanding of a concept. Furthermore, it is argued that students who learn mathematics through a wide range of complex problems 'outperform other students whose learning is more compartmentalized and abstract.' (McREL, 2012, p.56) In life, real problems are not divided into various mathematical strands. Therefore, presenting mathematical problems in real life contexts encourage students to take ownership for their learning.

Confucius, a Chinese philosopher (551 BC-479 BC) promoted the idea that students learn best by doing; 'I hear and I forget; I see and I remember; I do and I understand.' When pupils learn using manipulatives, various senses are brought into play with the brain making a greater number of connections to new information. Essentially, 'the more connections that are made, the better a learner can understand an idea.' (Sutton and Krueger, 2002, p.90)

One area of Mathematics that could benefit from this approach is the GCSE topic, 'Constructing Simple Loci.' From my personal experience of studying GCSE Mathematics, this is often a topic that is dealt with superficially. When taught, it demands the pupils to remember a series of steps in order to answer questions without understanding the processes.

In this project, I set out to create a hands-on 'Loci-kit' resource to help pupils to understand the concept of Loci and determine the place of the locus of points about a fixed point, equidistant from two points, equidistant from two non-perpendicular lines, a fixed distance from a line and a fixed distance from a 2-D shape. The Loci-kit resource also incorporates the use of Prezi, see Appendix 1, which serves as an online platform to guide pupils through real-life problems and direct them onto further GCSE type questions.

Literature Review

This literature review will consider the difficulties faced by pupils in the topic of Loci, including pupil misconceptions with the topic and the methods used to support teaching this topic. This literature will also consider the educational theory behind mathematical learning and the theory considered most appropriate to introduce the concept of Loci using practical methods. Furthermore, this study will examine the effectiveness of incorporating hands-on learning resources and real life contexts to teach Loci. All of this will support the creation of a Loci-Kit resource that will accompany this investigation, ensuring that the resource addresses the initial difficulties and misconceptions with Loci, as well as ensuring the resource is accessible and engaging to pupils in a GCSE Mathematics class.

Educational Theory

'Those who can, do. Those who understand, teach.' (Shulman, 1986, p.14)

Pedagogical Content Knowledge

Despite a teacher's deep understanding of a subject area, he or she must be able to foster an understanding of the subject or concepts for students. In other words, they must have an understanding of what Shulman describes as 'Pedagogical Content Knowledge.'

Pedagogical Content Knowledge goes beyond the knowledge of the subject matter to the dimension of subject matter for teaching that 'embodies the aspects of content most germane to its teachability.' (Shulman, 1987, p.9) It includes an understanding of what makes the learning of specific topics easy or difficult, conceptions and preconceptions that students of different ages and backgrounds bring with them to learning.

A recent report on the teaching and learning of geometry by the Royal Society and Joint Mathematical Council (2001) argues that, 'the most significant contribution to improvements in geometry teaching will be made by the development of good models of pedagogy, supported by carefully designed activities and resources.'

Theory and approach to teaching and learning Mathematics
There are three critical components which Shellard and Moyer (2002) believe are key to effective mathematics instruction: 'teaching for conceptual understanding, developing children's procedural literacy and promoting strategic competence through meaningful problem-solving investigations.'

There are two prevalent approaches to Mathematics instruction. In skills-based instruction, which is a more traditional approach to teaching mathematics, 'teachers focus exclusively on developing computational skills and quick recall of facts' whereas in concepts-based instruction, 'teachers encourage students to solve a problem in a way that is meaningful to them and to explain how they solved the problem.' (www.andrews.edu) In an effective mathematics classroom, the teacher should facilitate learning by posing challenging and interesting questions to the pupils. This will spark the curiosity of the pupil and encourage them to investigate further. In regards to the students, Protheroe (2007) believes that students must be 'actively engaged in doing mathematics' and should be 'solving challenging problems'. It is essential that students can see that Mathematics is not a stagnant field of textbook problems; rather, it is a dynamic way of constructing meaning about the world around us. Most researchers agree that both approaches to mathematics instruction in the classroom are important, as teachers should strive towards skill based learning that is meaningful to the pupil.

Using multiple representations to communicate mathematical ideas is beneficial to learning. This finding agrees with the research works of Raphael and Wahlstrom (1989) that 'students who use manipulatives in their mathematics classes usually outperform those who do not.' This will essentially provide students with a bridge from the 'concrete understandings of mathematics' that have been learned at KS3 level to the 'abstract understandings' that will be required of them at KS4 level. (www.andrews.edu)

Introducing a new topic in Mathematics
Ma and Papanastasiou (2006) believes that the choice of an instructional method for introducing a new topic in Mathematics is largely based on 'traditional wisdom or instructional convenience, rather than working knowledge derived from empirical evidence of research studies.' One of the strands of instruction classified in mathematics is 'Cognitively Guided Instruction' (CGI) which is based on four interlocking principles: 'teacher knowledge of how mathematical content is learned by their students; problem solving on the focus of instruction; teacher access to how students are thinking about specific problems; and teacher decision making based on teachers knowing how their students are thinking.' (Secada, 1992 cited in Ma and Papanastasiou, 2006) Therefore, this allows teachers the flexibility to engage students in learning, based on their knowledge of their student's thinking processes.

Ma and Papanastasiou (2006) carried out an investigation that measured the effect of various instructional methods on academic achievement when introducing a new topic in mathematics. The investigation considered student performance in different mathematical areas including: a) Mathematics as a whole, b) Algebra, c) Data Analysis, d) Fraction, e) Geometry and f) Measurement.

The table below presents the effects of using different instructional methods on academic achievement in mathematics as a whole.

Table 1. Different instructional methods of beginning a new topic in mathematics in relation to student performance in mathematics (Ma & Papanastasiou, 2006).

Instructional methods	*Teacher instructional effect*	*Variation in teacher instructional effect*	*School mean instructional effect*
Having the teacher explain the rules and definitions	-0.003	0.001	0.034
Discussing a practical or story problem related to everyday life	0.065***	0.002*	-0.072
Working together in pairs or small groups on a problem project	0.121***	0.005***	0.001
Having the teacher ask us what we know related to the new topic	0.046***	0.002***	0.096
Looking at the textbook while the teacher talks about it	0.033***	0.002***	-0.120
Trying to solve an example related to the new topic	-0.019**	0.001	-0.308*

As evident from the table above, four instructional methods with statistically significant (positive) effects on overall mathematics achievement were 'discussing a practical or story problem related to everyday life; working together in pairs or small groups on a problem project; having the teacher ask us what we know related to the new topic and looking at the textbook while the teacher talks about it.' Only one instructional method, 'trying to solve an example related to a new topic,' showed significant (negative) effects on academic achievement in mathematics.

Furthermore, the various instructional methods were employed when introducing a new topic in other mathematical areas. In the table below, the results show the effect on academic achievement in the area of Geometry.

Table 2. Different instructional methods of beginning a new topic in mathematics in relation to student performance in geometry (Ma & Papanastasiou, 2006).

Instructional methods	*Teacher instructional effect*	*Variation in teacher instructional effect*	*School mean instructional effect*
Having the teacher explain the rules and definitions	0.002	0.001	0.072
Discussing a practical or story problem related to everyday life	0.078***	0.002***	-0.153
Working together in pairs or small groups on a problem project	0.123***	0.007***	-0.067
Having the teacher ask us what we know related to the new topic	0.054***	0.001	0.066
Looking at the textbook while the teacher talks about it	0.072***	0.005***	-0.152*
Trying to solve an example related to the new topic	-0.005	0.001	-0.307**

The research found that the instructional methods where 'teachers discuss a practical or story problem related to everyday life' and where 'students work together or in small groups on a problem project' had statistically significant positive effects on student geometry performance.

This would imply that a resource should therefore employ both of the instructional methods mentioned above in order to increase student performance and understanding of the concept of constructing simple Loci. This acts in contrast to ‘having the teacher explain the rules and definitions’ which had little instructional effect on mathematics learning in the investigation. Ma and Papanastasiou (2006, p.12) believe that their findings indicate that using different instructional methods to introduce a topic in mathematics has different impacts on students’ mathematics performance. Consequently, as a teacher it is vital to consider how we introduce a new topic in mathematics as this will contribute to student learning and achievement.

In addition to this, ‘working together in pairs or small groups on a problem project’ had the single largest positive instructional effect across the five areas of mathematics that were considered. In terms of Geometry, the positive result of 0.123 suggests that student-centred, co-operative learning is more suitable than teacher-led instruction when learning a new topic, such as Loci.

Learning

In order to plan and deliver an effective course, it is vital to recognise who your students are as learners. Brookfield (2006) states that some teachers may exhibit *‘an admirable command of content’* and possess *‘a dazzling variety of pedagogical skills’* but without knowing what is going on in their students’ heads, this knowledge and skill may be *‘exercised in a vacuum of misunderstanding.’*

What is learning?

> Learning involves ongoing, active processes of inquiry, engagement and participation in the world around us
>
> (Bransford, Brown and Cocking, 2000).

Regardless of ability, everyone has the potential to grow and learn through new and exciting opportunities. Learning was once understood as a linear process, a progression through different ages and stages. However, in more recent times, researchers and educators view ‘growth development and learning as a more dynamic system.’ (Fischer and Heikkinen, 2010, p.260)

Mainstream classrooms are becoming increasingly diverse in language and specific learning needs and so there is a need to implement different learning strategies if we are to reach the students in present classrooms. A principal of Howard Gardner’s Theory of Multiple Intelligences (MI) is that people learn, represent and utilise knowledge in a variety of different ways such as visually, kinesthetically or logically. These differences challenge an education system that assumes everyone can learn the same materials in the same way and be tested through a universal measure. According to Gardner (1991), students and society would be better served if ‘disciplines could be presented in a number of ways and learning could be accessed through a variety of means.’

Hands-on learning in Mathematics

Mathematical learning in young children is strongly linked to sense perception and concrete experience which will lead to the understanding of abstract concepts. Sutton and Krueger (2002, p.90) believe that, ‘the more avenues there are to receive data through the senses, the more connections the brain will make. The more connections that are made, the better a learner can understand an idea.’ The human brain appears to be highly interconnected and, like a classroom, complex and multi-dimensional. Through the development of new brain learning models and cognitive models of learning, various strategies can be examined as to how students learn.

One such pedagogy is the experiential value of 'hands-on' learning, recognised as significant in engaging students. Mathematics is often thought of as a subject that relies on memorisation of facts and practising skills. However, true success comes when a pupil must answer correctly but cannot remember a fact or has forgotten a skill. Hands-on learning drives authentic understanding and application as opposed to memorisation algorithms or 'tricks'.

Satterwait (2010, p.8) has identified three factors that play a significant role in hands-on practice: 'the influence of co-operative learning and social constructivist understandings; mediated learning through the use of objects; and embodiment as a way of students gaining understanding and making meaning of their experiences.'

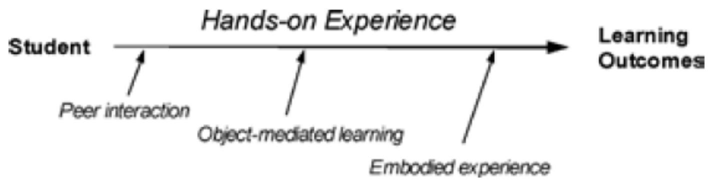

Figure 1. The three factors that Satterwait (2010, p.8) has identified play a significant role in hands-on practice.

The use of manipulatives is essential and effective in both primary and post-primary institutions as it is key that all students need to approach the learning of mathematics by actively doing mathematics. Sutton and Krueger (2002) believe that 'when students can touch and move objects to make visual representation of mathematical concepts, different learning modalities are addressed.' This will lead to conceptual understanding and therefore will enhance cognitive development of children.

Teaching Mathematics in real life contexts

> In real-life, learning experiences are not separated into academic disciplines or subject areas. A student's classroom experience should mirror this
>
> (Sutton and Krueger, 2002).

As early as 1939, John Dewey warned that isolation in all forms is to be avoided and that we should strive to bridge the 'gap existing between the everyday experiences of the child and the isolated material supplied in such large measure in the school.' (1956, p.75) This study attempts to build a bridge between the pupil's everyday experiences and the concept of Loci. In recent years, school mathematics has shifted from a fixed body of knowledge towards a more student-centered classroom, which focuses on higher-order thinking skills, problem solving and real world contexts. Using a context with what students are already familiar, amplifies student's confidence and comfort with the content being taught.

Children at the 'formal operational' stage of development identified by Piaget, are capable of forming hypotheses and deducing possible consequences, allowing the child to construct his own mathematics. Reasoning skills within this stage include the 'application' process that involves 'students connecting mathematical concepts to real-life situations.' (Ojose, 2008, p.28) Pupils can build a strong foundation of knowledge based on their physical experiences and meaningful contexts. Sutton and Krueger (2002) believe that 'the more senses used in instruction, the better learners will connect the information in their memories.' Therefore, it is believed that students are perhaps more likely to take ownership and determine direction for their own learning.

Constructing simple Loci in the GCSE classroom

Concept of Loci in Mathematics

'Locus' is a Latin word which means 'place.' (Robert and Glenn, 1992) A locus of points is a set of points, and only those points that satisfy a given condition. A locus can be a straight line, curved line, area or region in two dimensions or a volume in three dimensions. 'Constructing simple Loci' is a topic in the T5 Unit of the CCEA GCSE Mathematics Curriculum. Pupils should be able to 'use ruler and compasses for standard constructions: construct Loci.' (www.rewardinglearning.org.uk) This includes using appropriate mathematical tools to construct the bisector of an angle, the locus of points equidistant from two points, the locus of points a set distance from a 2D shape and the locus of points equidistant from a line. A further requirement at GCSE is to construct the graphs of simple Loci, including the region bounded by a circle and an intersecting line.

Where is Loci used in everyday life?

There are many opportunities for pupils to link their learning in Mathematics to real life examples. In the YouTube video titled 'KS4 Maths4Real Loci, Duncan (2014) presents an insight into the relationship between Loci and everyday life. Circular Loci are the basis of mobile phone technology. For example, to find the nearest petrol station in an area.

Furthermore, Loci can be related to guide dog training centers where the dog's ability to judge and maintain distances is crucial. A basic skill is to walk along a pavement, keeping equidistant from buildings and the edge of the kerb to avoid the person colliding with an obstacle or stepping onto the road. In the video, the dog represents the resulting path and shows that it can follow the same rule between two converging lines, bisecting the angle in the process.

In terms of house hunting, an actress on the set of Emmerdale is looking for a property. The ideal location would be halfway between the studios and the Emmerdale village. Constructing the perpendicular bisector of these two locations on a map of Leeds would help her to see the likely properties in the vicinity of her line. (www.youtube.com)

Misconceptions around Loci

Students are often taught to solve simple locus problems under the superficial impression, 'to solve this locus problem, construct the geometrical figure.' On one occasion to find the locus of points equidistant from two given points, a student was overheard saying, 'you just find the two points and just join them up.' (Cha and Noss, 2001, p.84)

The poor performance in mathematics and in particular Loci, has been attributed to several factors including 'poor teaching methods, lack of teaching and learning resources, and the abstract nature of mathematics.' (Githua and Mwangi, 2013) In an attempt to seek a teaching strategy that increases mathematics achievement, the effectiveness of Loci-kit models was investigated in secondary schools in Kenya. This was intended to combat the fact that, 'learners find it difficult to grasp concepts in loci while a majority of teachers dislike teaching the topic.' (Salmon, 2005)
When questioned as to why geometrical mistakes are made when constructing Loci, a student said that they 'underestimated drawings so when they saw it in an exam they panicked and could not do it properly.' (Özerem, 2012, p.27) Consequently, Salman proposed the use of relevant concrete materials in the teaching of mathematics in which students 'participate and interact with models and manipulates in order to promote meaningful understanding of mathematics concepts among learners.' (2009, cited in Guthua and Mwangi, 2013)

Student's thinking consists of many different things. One problem that leads onto other learning difficulties in mathematics is the misconceptions that students may have from 'previous inadequate teaching, informal thinking or poor remembrance.' (Allen, 2007, p.1) From the Encarta Online

Dictionary, a misconception is 'a mistaken idea or view resulting from a misunderstanding of something.' Furthermore, misconceptions in geometry are often related to a student's perception of shape and three dimensions. The Department for Education and Skills (2006, p.24) believe that the work on Loci is 'embedded with mental mathematics because the imagery is vital in helping pupils to predict and, later, check solutions.' Perhaps a constraint, when Loci is produced on paper, is that the student must require a good understanding of the reasoning and constructions in geometry. It is suggested that all students spend time learning to 'recognize the properties of shapes, through construction and testing hypotheses about the paths of moving points.' (Education and Skills, 2006) This will increase their confidence and competence in solving problems that are more complex.

Misunderstanding the question and the associated terminology is an obstacle that many students face. For example, Özerem (2012) highlighted that pupils 'could not draw the angle bisector.' The student was expected to draw the angle bisector of the given angle but instead s/he constructed just the angle. The use of proper terminology when introducing terms in geometry is essential in a pupil's understanding. Moreover, the student is expected to understand the meaning of the terms perpendicular bisector, equidistant, locus of fixed point etc. in order to succeed in geometry learning. Reasoning as to why this is the case should be based on their previous knowledge of the properties of 2D shapes including circles, rhombus and kites.

Hands-on Learning in Loci

Chester, Davis and Reglin (1991, cited in Githua and Mwangi, 2013) found that 'third grade students who were presented geometry concepts with manipulatives scored significantly higher on the posttest than the group that was presented concepts using only drawings and diagrams.'

In Kenya, Mathematics has consistently been ranked last in performance in comparison with other subjects offered. A baseline survey research carried out by Makueni District in 2007 revealed 'the mathematics topic Loci was rated as the most challenging topic after the topic of linear inequalities.' (Kibui and Macrae, 2005 cited in Githua and Mwangi, 2013) As a result of the findings, the researchers constructed Loci-Kit models to augment the teaching of the topic Loci to secondary school students. For example, a model of a wall-clock, model of a paintbrush representing a line and a model of a goat tethered in a grazing field.

The researchers found that students who were taught by use of Loci-Kit models achieved significantly higher scores than those taught using conventional methods. Again, this finding agrees with the research works of Raphael and Wahlstrom (1989) that 'students who use manipulatives in their mathematics classes usually outperform those who do not.'

What makes a good learning resource in Mathematics?

Properties of a good Hands-on Learning Resource

In order to create a good hands-on learning resource in the classroom, the materials should be used effectively. According to Suydam and Higgins, (1976, p.92 cited in Post, 1981) manipulative materials should be 'used in ways appropriate to Mathematics content; used in conjunction with other aids and that the simplest possible materials should be employed.' Mathematics manipulatives do not teach on their own, rather they are open pathways to learning.

Prior to designing a learning resource, it is vital to consider how to reach out to pupils of different learning styles. Some pupils find even basic concepts in Mathematics difficult to grasp and it can be a challenge to make lessons accessible to all. In a development of special interest to teachers of mathematics, Silver, Thomas and Perini (2003, cited in Silver, Brunsting and Walsh 2008) applied the research on learning styles specifically to the study of mathematics. Out of their work came the identification of four distinct Mathematical learning styles as shown below.

Table 3. The four types of mathematics students (Silver, H.F., Brunsting, J.R., Walsh, T.,2008)

The Four Types of Mathematics Students	
Mastery Math Students . . .	**Interpersonal Math Students . . .**
Want to . . . learn practical information and set procedures. *Like math problems that* . . . are like problems they have solved before and that use algorithms to produce a single solution. *Approach problem solving* . . . in a step-by-step manner. *Experience difficulty when* . . . math becomes too abstract or when faced with non-routine problems. *Want a math teacher who* . . . models new skills, allows time for practice, and builds in feedback and coaching sessions.	*Want to* . . . learn math through dialogue, collaboration, and cooperative learning. *Like math problems that* . . . focus on real-world applications and on how math helps people. *Approach problem solving* . . . as an open discussion among a community of problem solvers. *Experience difficulty when* . . . instruction focuses on independent seatwork or when what they are learning seems to lack real-world application. *Want a math teacher who* . . . pays attention to their successes and struggles in math.
Understanding Math Students . . .	**Self-Expressive Math Students . . .**
Want to . . . understand why the math they learn works. *Like math problems that* . . . ask them to explain, prove, or take a position. *Approach problem solving* . . . by looking for patterns and identifying hidden questions. *Experience difficulty when* . . . there is a focus on the social environment of the classroom (e.g., on collaboration and cooperative problem solving). *Want a math teacher who* . . . challenges them to think and who lets them explain their thinking.	*Want to* . . . use their imagination to explore mathematical ideas. *Like math problems that* . . . are non-routine, project-like in nature, and that allow them to think "outside the box." *Approach problem solving* . . . by visualizing the problem, generating possible solutions, and exploring among the alternatives. *Experience difficulty when* . . . math instruction is focused on drill and practice and rote problem solving. *Want a math teacher who* . . . invites imagination and creative problem solving into the math classroom.

Therefore, the resource needs to incorporate problem solving, real-world applications and imaginative thinking in order to be accessible to at least two out of four types of learners above.

In addition to this, it is important to consider whether the learning resource will be used individually, in small groups or as a whole class activity. The teacher should play the role of the facilitator while the pupils are using the resource, prompting the pupils with key questions as to why they believe they have answered the question correctly. It is important that both teachers and pupils express key language so that the pupils become more familiar with the terms and meanings.

Use of ICT in a Mathematics resource
The NNS (DfEE, 1999 cited in Drews, 2007) advocates the use of ICT resources, providing that 'it is the most efficient and effective way to meet your lesson objectives.' The use of ICT can be implemented in many ways in the Post-Primary Mathematics classroom using PowerPoint, Prezi, Movie Maker and Podcasts etc. Part of the Loci-Kit resource will be Prezi based. Prezi was chosen based on its accessibility for classroom and its ability to direct the learners to further GCSE type questions where they can put their knowledge of each concept into practice. According to Higgins and Muijs, (1999, cited in Drews, 2007) 'more explicit links need to be made between computer activities and other planned activities in order that pupils develop a greater awareness of Mathematical connections.'

Methodology

Planning and Creation
Prior to creating the resource, it is important to consider what format the resource will take, what loci content should be explored through the resource, how the resource should be presented in a real life context, and what materials should be used.

The resource is intended to be used to aid the understanding of the main concepts of Loci using real life examples. Initially, there would be five different scenarios to represent the five different concepts of Loci that are included in the GCSE Mathematics specification. Having previously been sketched on paper, it appeared that each scenario would be transformed into a 3-D format to align with the focus on 'hands-on' learning.

The aim of this project is to create a resource that consists of several elements; a hands-on learning board where there will be five real life scenarios, each reflecting a specific aspect of Loci, and a Prezi Presentation, '5 Loci in Locustown' which will guide both pupils and teachers around the tasks. Originally, each scenario would be collated onto one board where pupils could progress around the board solving each individual Loci problem. The board would be landmarked with five scenarios that are realistic in an everyday town setting. However, it became evident that designing on one board would be unrealistic and impractical in regards to the scaling of each question. In addition to this, the collated design could be potentially distracting to pupils and closed in the sense that there would be limited opportunity for extension work.

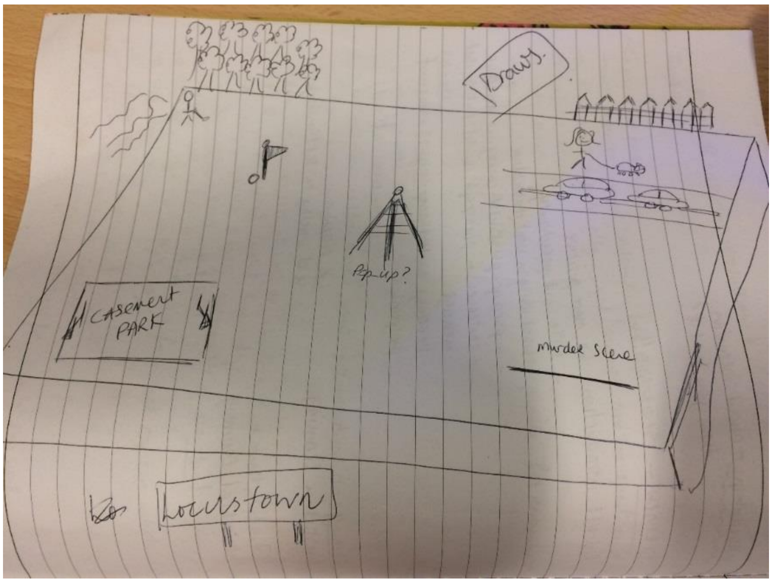

Figure 2. Original idea of a 'Locustown' board.
Therefore, the remodelled five-board Loci kit, each addresses a specific area of Loci in the GCSE curriculum. Each board will test the pupils' understanding of language in each of the scenarios and the ability to satisfy the conditions of the question and successfully place the counters as the locus of points. This should essentially create a path and reinforce the definition of Loci in the eyes of the pupil. The Loci-kit can act as a stimulus for the pupils, prompting them to think of problems in real life context so that the mathematics will have meaning and relevance. This is especially useful when introducing a new topic at GCSE level.

The interactive 'Prezi' provided an online platform for the pupils in bridging the gap between virtual and hands-on learning. It also emphasised that although each concept of Loci is different, the fundamental principle of Loci still holds; 'a locus is a path, formed by a point which moves according to some rule.' Furthermore, GCSE type questions, selected through CCEA's Topic Tracker and

Edexcel's Past Paper Forum, were available coupled with the added support of a step-by-step Success Criteria.
Therefore, this resource is hands-on and incorporates ICT which mirrors the view of The Royal Society (2001), who recommend that the geometry curriculum be developed 'to give greater emphasis to work in 3-dimensions and to make better use of Information and Communication Technology.'

The choice of materials used in the creation of the resource is important in regards to practicality. Initially, each board in the Loci-kit was designed on a wooden base and transported in a drawer-type holder. For example, the use of MDF wood as it is lightweight and relatively inexpensive.

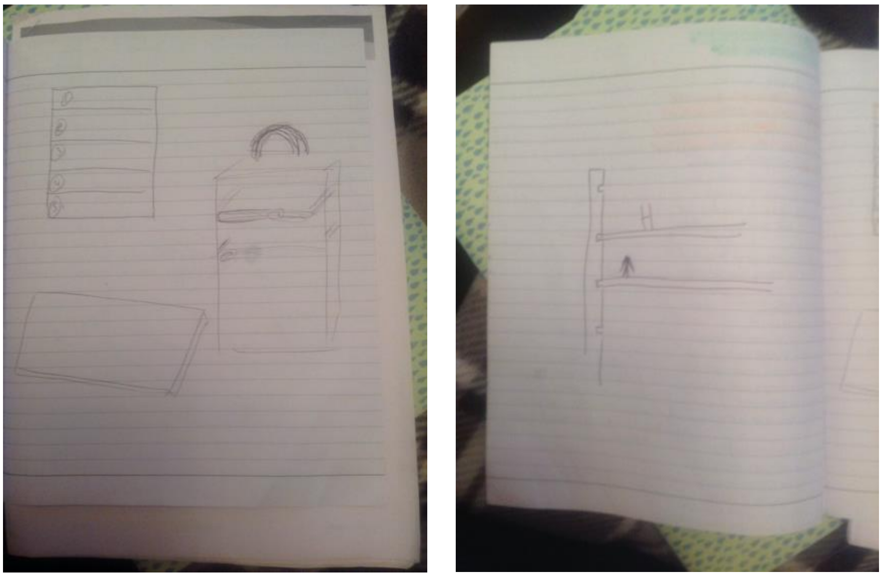

Figure 3. Original design for wooden drawer system.

However, the choice of materials used for the boards changed from wood to an A4 clipboard, based on the fact it was designed using PVC covered card that is strong and durable, yet light and practical. Furthermore, this material is very cost effective in creating the resource. In order for each scenario to stand out and appear realistic, felt and foam sheets, artificial grass and lollipop sticks were some of the materials that were used to set each scene. When glued to the board, these materials stood out, making learning accessible to those who are visually inclined and enjoy using their sense of touch.

In order for the resource to be hands-on, pupils will use counters in order to physically show their understanding of Loci as a path on each board. This will be successful if the counters are placed correctly in accordance to the condition of the Loci and scaling of the question. The finished object will show how placing each of the counters as points on the board will potentially plot a path of points, which reinforces the pupil's understanding of the definition of Loci; a locus is a path, formed by a point, which moves according to some rule.

In order to assist the pupil's understanding and to address the 'why-ness' of the underlying Mathematics, the resource can be manipulated by laying acetate paper on the surface of the boards. The properties of 2-D shapes such as the circle, kite and rhombus proves why the locus of points behaves according to a specific rule. For example, to bisect the angle or to perpendicularly bisect a line. Furthermore, the sheets of acetate are designed to direct pupils, step-by-step, through the construction of Loci by accurately constructing the lines and arcs using a compass with the assistance of the success criteria in the Prezi.

Trial and research

In order to test the effectiveness of this resource in a classroom, it was decided to trial with two Mathematics teachers, who have experience of teaching GCSE, a set of four undergraduate teachers and two Year 12 GCSE Mathematics pupils. The respondents were chosen based on the desire to gain honest opinions from three groups of people whose experience differs; two senior maths teachers who have taught Loci at GCSE in the past, four undergraduate teachers who will be teaching Mathematics at GCSE level in the future and two GCSE pupils who will be covering the topic this year. The main form of data collected for the analysis of this resource was qualitative, as this method of data collection allows for personal opinions and constructive feedback, which is necessary when considering the use and practicality of using this resource in a GCSE Mathematics classroom. However, quantitative data were also collected through set questionnaires.

The questionnaire was used to determine the general opinions of the respondents in relation to using this resource in the classroom. Questionnaires are useful in the sense that they target a larger audience and can be distributed online. Furthermore, they allow the audience to give a quick response to the questions. This research used web-based questionnaires as the method of distribution as it is much easier to complete, collect and analyse the data electronically.

The web-based questionnaire, designed through Survey Monkey for this research, was comprised of six closed questions where the teachers and pupils could rate the effectiveness of using the Loci-kit resource to teach Loci. Three of the questions were based on a rating scale whilst the other three of the questions were multiple-choice type, which may ask for an elaborate comment if a particular answer is selected. The web-based questionnaire examined questions relating to; the effectiveness and accessibility of the resource and how well it addresses the Mathematical content in Loci. (See Appendix 1.).

In addition to the questionnaires, the interviewees were then asked to attend a semi-structured interview. This was an effective way of collecting qualitative data by allowing explanations to be drawn. Furthermore, semi structured interviews also provided the opportunity to express personal ideas and opinions of the resource and its potential use in the classroom.

The semi-structured interview consisted of open-ended questions. The questions were similar in nature to those asked in the questionnaire, but essentially allowed more feedback to be obtained. The interview questions were designed to target the effectiveness of the resource in regards to hands-on learning, the mathematical content covered and the potential difficulties and limitations that may arise in the GSCE classroom. In addition to this, the interviewee could provide further feedback on the resource as a whole and perhaps comment on something that the researcher had not considered.

Ultimately, the main purpose of using a web-based questionnaire followed by a set of semi-structured interview questions was to evaluate the effectiveness of incorporating hands-on learning into a GCSE Mathematics lesson. The results will draw conclusions as to how effective the hands on Loci-kit resource was when introducing and teaching the concept of Constructing simple Loci.

Analysis and evaluation

This analysis and evaluation will consider the responses from the web based questionnaire and the semi-structured interviews and also use the results from the quantitative and qualitative data that are produced. These results will be analysed and conclusions will be drawn as to how effective this resource is when used in a GCSE Mathematics class. The resource will then be evaluated in reflection of the analysis. This shall include how effective the resource is, the advantages of incorporating this type of learning at GCSE level and its limitations.

Analysis of questionnaire

From the results of the questionnaire, it is evident that the Loci-kit resource is highly effective in regards to it being enjoyable, engaging and stimulating. Each of these aspects scored an average result of 8.9 out of 10. In relation to the how effective using the Loci-kit is in supporting the learning of the Loci at GCSE, seven out of eight people said that the resource supports it very well. One respondent answered 'somewhat well' which suggests that perhaps certain aspects of the resource were not suitable when teaching constructions. In relation to the incorporation of mathematics in the resource, it was also rated as highly effective in the areas of; language, real life context, definition and construction. The lowest average score was for construction, which was scored 8.5 out of 10. This suggests that perhaps the resource assumes pupil competence in using a compass. Furthermore, additional support may be needed to ensure pupils understand the mathematics behind the constructions. This mirrors the research, suggesting that all students should spend time learning to 'recognize the properties of shapes, through construction and testing hypotheses about the paths of moving points.' (Education and Skills, 2006) The highest average score was for the real life context of the mathematics which scored 9.4 out of 10, suggesting that teachers and pupils found the contextualised scenarios in 'Locustown' effective in supporting the teaching of Loci.

All of the responses to the questionnaire identified the hands-on learning aspect to be the best feature of the resource as it supported the practicality of constructing simple Loci. As aforementioned in the Literature review, 'students who were presented geometry concepts with manipulatives scored significantly higher in the posttest than the group that was presented concepts using only drawings and diagrams.' (Chester, Davis and Reglin, 1991, cited in Githua and Mwangi, 2013) Five out of eight respondents also believed that the use of ICT using Prezi stood out as one of the best features, which reflects the belief that using ICT 'is the most efficient and effective way to meet your lesson objectives.' (DfEE, 1999 cited in Drews, 2007) None of the respondents identified 'appearance' as being the best feature of the resource, something that was given a lot of thought during creation. This suggests that the ultimate aim of the resource is to effectively teach the mathematical content of Loci using various strategies. This reiterates the belief that 'when students can touch and move objects to make visual representation of mathematical concepts, different learning modalities are addressed.' (Sutton and Krueger, 2002) Consequently, this leads to conceptual understanding and enhancing the cognitive development of children.

The majority of responses indicated that the online Prezi presentation is very accessible, with an average score of 9.3 out of 10. Two of the responses scored eight out of 10, suggesting that there was some difficulty in accessing the resource online, possibly due to internet connectivity.

Overall, seven out of eight people who were surveyed stated that they would use the Loci-kit resource in their GCSE classroom, commenting that; the resource is a fresh and innovative way to introduce a new topic, it is engaging to various types of learners by using both hands-on learning and ICT, and that it is successful in creating a link between mathematics and real life. This imitates the research that 'the most significant contribution to improvements in geometry teaching will be made by the development of good models of pedagogy, supported by carefully designed activities and resources.' (The Royal Society and Joint Mathematical Council, 2001) The respondent who stated that they would not use it in the classroom, commented that it would be unrealistic to implement this at GSCE due to time constraints and a tight post-primary timetable.

Analysis of interviews
The senior mathematics teachers and one of the undergraduate mathematics teachers who completed the web-based questionnaire also took part in the semi-structured interview questions that followed. The set of interview questions enables more qualitative data to be collected and to gain a greater insight into the dimensions of the resource.

In relation to which aspects of the resource made it enjoyable, engaging and stimulating, the respondents noted the success of the format of the resource. Comments included that the hands-on learning boards were effective as the pupils placed the counters based on how they interpreted and solved the problem. This reflects the research on learning styles based on Silver, Thomas and Perini, that the resource needed to incorporate 'problem solving, real world applications and imaginative thinking' in order to be accessible to at least two out of the 'four different types of mathematics students'. (2003, cited in Silver, Brunsting and Walsh, 2008) In addition to this, the respondents stated that the use of ICT and manipulatives enhanced the learning experience for pupils when faced with a Loci problem. This fulfilled a principal of Gardner's Theory of Multiple Intelligences (MI) that people learn, represent and utilise knowledge in a variety of different ways and that if 'disciplines could be presented in a number of different ways, then learning could be accessed through a variety of means' (1991).

The interviewees also commented on the stimulating nature of the various real life problems, where essentially the mathematics was 'well-hidden' but effective in that the pupils would not realise that they were learning. This feedback mirrors the evidence driven from Ma and Papanastasiou's investigation (2006, p.12) that the instructional methods where 'teachers discuss a practical or story problem related to everyday life' had statistically significant (positive) effects on student geometry performance.

The interviewees were then asked to identify any areas of the mathematics that were not covered effectively in the resource. The respondents found no faults with language but noted that perhaps greater learning support could be given to pupils who struggle with Literacy and Mathematics. This could be in the form of definition buttons on Prezi. This type of observation reflects the research, that the use of proper terminology when introducing geometrical terms is essential in a pupil's understanding. For example, Ozerem (2012) highlighted that pupils 'could not draw the angle bisector' as they did not understand the word 'bisect'. Furthermore, a refreshment lesson on the properties of 2-D shapes was suggested to reveal the hidden mathematics behind the constructions.

In relation to the style of the problem questions, the respondents commented that the real life context provided good engagement for the pupils. This is evident in the 'formal operational' stage of development, identified by Piaget, where pupils apply their reasoning skills to 'connect mathematics to real-life situations.' (Ojose, 2008, p.28) Consequently, mathematics has meaning to the pupils, something which is absent from standardised textbook problems. In addition to this, the Prezi presentation worked in harmony with the boards, making the resource more interactive.

When asked about the aspects of definition, the interviewees noted that this resource was a prime example of the teacher as the facilitator. The five locus landmarks in the Prezi presentation assisted the pupils in composing a working definition of what Loci was i.e. the pupils plotting the counters on the board revealed the first part of the definition, 'a locus as a set of points in a path...' Consequently, by following the conditions of the question they follow with the end of the definition, '…which satisfy a certain condition.' This type of discovery learning is key for pupil understanding in comparison to 'having the teacher explain the rules and definitions,' which was found to have a significant (negative) effect on pupil's academic achievement when introducing a new topic in Mathematics. (Ma and Papanastasiou, 2006)

In regards to construction of simple Loci, the respondents agreed that the acetate sheets were simple and effective when placed on top of the boards to show pupils if they had placed the counters correctly. This reflects Suydam and Higgins' belief that materials should be 'used in ways appropriate to Mathematics content; used in conjunction with other aids and that the simplest possible materials should be employed.' (1976, p.92 cited in Post, 1981) This, coupled with the success criteria for the GCSE questions, made it accessible for the pupils. However, one respondent noted that competence in using a compass prior to constructing would be essential. Furthermore, perhaps another acetate layer would have been beneficial to link the pupil's previous knowledge of the properties of 2-D shapes to the constructions.

When asked what aspects of the resource would encourage teachers to use it in the classroom, the interviewees identified that the Loci-kit was an effective way of introducing the new topic, as using multiple representations to communicate mathematical ideas is beneficial to learning. This harmonises with the research works of Raphael and Wahlstrom (1989) that, 'students who use manipulatives in their learning usually outperform those who do not.' One respondent commented that the resource enabled pupils to integrate their own ideas by working through relatable problems. This avoids the pupils memorising a set of instructions with minimal understanding. These were regarded as the strengths of the resource.

On the contrary, when asked to identify the weaknesses of the resource, one of the interviewees stated that the counters used were too small and hard to handle. The respondents also questioned whether the resource could realistically be used in the GCSE classroom, mainly centring on time management issues and timetabling in a post-primary setting. Furthermore, one respondent questioned whether it would be better if the entire resource were online.

When asked what improvements could be made to the teacher guide, the interviewees stated that perhaps more in-depth instruction would be helpful in regards to the underlying mathematics. This is vital as The Department of Education and Skills believes that the work on Loci is 'embedded with mental mathematics because the imagery is vital in helping pupils to predict and, later, check solutions.' (2006, p.24) Potentially, this would overcome geometrical misconceptions, which are often related to a student's perception of shape and dimension.

Finally, the respondents had the opportunity to make additional comments on the resource and its potential use in the classroom. One respondent noted how one of the problem questions could be made more cross-curricular by changing 'Gaelic' Football to football. In addition, it was noted that this resource would be of greater support to weaker GCSE pupils and that extension questions could be included in the Prezi. This suggests that the resource may need to be altered if it was to be used with mixed ability classes and those in different denominational schools.

Evaluation of Advantages

From the analysis of the web-based questionnaire and interview, it is evident that there are many advantages to using the resource created for this project to aid the teaching of Loci. The resource is successful in engaging and stimulating the pupils through a variety of means. This will make the pupils more likely to develop a greater understanding of the topic and how to accurately construct Loci in an exam. The real-life context of the questions and the hands-on learning element will appeal to pupils who perhaps dislike mathematics and withdraw themselves in class when faced with textbook questions. Therefore, the success of this resource is dependent on the ability to avoid isolation in all forms and strive to 'bridge the gap existing between the everyday experiences of the child and the isolated material supplied in such large measure in the school.' (Dewey, 1956, p.75)

The resource is unique in its approach to the topic of Loci. It introduces the topic in a way that many have not considered, using hands-on learning coupled with ICT to engage pupils in purposeful learning. This reflects concepts-based instruction where 'teachers encourage students to solve a problem in a way that is meaningful to them.' (www.andrews.edu) For example in the Blind dog scenario, the Loci was constructed and identified as the perpendicular bisector of the two points, which could be seen as a logical explanation for keeping the blind lady safe.

Evaluation of Limitations

Many improvements could have been made to this project had there been more time allocated to the creation of the resource. The resource could have been further developed to ensure that mathematical learning takes place and the pupils have a full understanding. Ideally, this resource could have been accompanied by 2-D shape models, which pupils could use to explain the mathematics behind the Loci.

A key limitation of this resource is that it has not been tested within a whole class setting, but rather with two GCSE Mathematics pupils. A classroom trial would have been more beneficial as it would better identify the strengths and weaknesses of implementing this in the classroom. This would have given a greater insight into how efficiently it could be used in a lesson and if the timing guidelines, set out in the teacher's guide, were realistic within a time-constraint lesson. The resource would have benefitted from trialling and it may be possible in the future.

A further limitation of this resource is that there is only one copy of each hands-on board at present. Despite being based on the effectiveness of hands-on learning, it could be redesigned as an entirely online learning program. This relates to the research that, 'more explicit links could be made between computer activities and other planned activities.' (Higgins and Muijs, 1999, cited in Drews, 2007) This could be in the form of interactive whiteboard pages where each scenario could be manipulated on screen with virtual counters. This would also be virtually pleasing to the pupils and easy to share online.

Conclusion

In conclusion, this research project was very interesting as it provided the opportunity to be creative in designing an innovative resource to teach the GCSE topic, Constructing Simple Loci.

An investigation into this topic, as to how pupils' understanding of Loci can be developed through hands-on learning and ICT, has proven to be very beneficial when implemented at GCSE level. Like many other mathematical topics, Loci can be exciting and engaging when taught through various means, and can be made relevant to pupils when problems are contextualised. Furthermore, it is important that an appropriate introduction to the topic be given for pupils to grasp the fundamental concepts before attempting a GCSE type question.

The key to success for the resource is the effective support of the class teacher. This is crucial, as the teacher is the mediator in assuring that the online Prezi presentation works effectively alongside the hands-on boards. This will ensure the pupils gain an understanding of the underlying mathematics within the topic of Loci.

The use of manipulatives at GCSE level are uncommon but should not be overlooked as support mechanisms, especially when introducing a new topic. This type of learning has proven to be beneficial to transitioning pupils, where the use of manipulatives 'bridge the gap from concrete understanding' learned at KS3 level to 'abstract understanding' at KS4 level. (www.andrews.edu) Furthermore, contextualised learning is fundamental to the resource, as each scenario in

'Locustown' has been designed in the interests of the pupil. Like any problem in life, whether it is maths-related or not, it will be of greater interest if it has meaning to you.

However, the importance of targeting underlying mathematics would have perhaps needed to be addressed further in this resource. This resource introduces and lays the foundations of the topic but could be developed to ensure that the pupils understand why, for example, angle bisection is linked to the properties of a kite. Similarly, understanding why perpendicular bisection is dependent upon the accuracy of constructing the arc of a circle. In light of this, perhaps the implementation of 2-D shape models would remedy any misconceptions the pupils may have in regards to this.

Using a hands-on learning resource coupled with ICT has proven to be very effective in the delivery of teaching the topic of Loci. The creation of the resource proved to be more complex than the initial planning, in ensuring that the mathematical content was covered within five realistic and age-appropriate problem scenarios.

There are certain aspects of the resource that could be improved upon if there was more time allocated to this research project. The product itself could have been trialled in numerous GCSE classrooms with pupils of differing abilities, backgrounds and interests in order to get a clearer picture of how successful this resource could be. Furthermore, a more in-depth resource could have been developed, with the addition of lesson plans, worksheets and extension materials to support the topic. Moreover, consideration should be given as to how to adapt the resource for pupils whose learning needs are obviously not met.

In conclusion, this research project and the accompanying resource effectively provides the opportunity for pupils to learn how to construct simple Loci in an open, creative and visual way. Teachers must ensure that the needs of all pupils are met through a variety of means when teaching such complex topics. Consequently, 'well-defined mathematical objectives, worthwhile and engaging mathematical tasks… and the appropriate use of hands-on activities to build conceptual understanding' should be implemented to meet the effective requirements of a GCSE mathematics lesson. (Larson, 2002) This will enable the pupil to master the concepts and skills required when constructing simple Loci.

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Appendix 1.